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PRESERVING THE FUNDAMENTAL FREQUENCIES OF BEAMS DESPITE MASS ATTACHMENTS

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1. INTRODUCTION

Recently, in the context of a project study it was considered as a subproblem to satisfy a design aim such that at least the fundamental frequency of a cantilever beam remains the same in spite of the addition of a mass at some point on the beam. After noticing that the conditions for the frequencies to remain the same in spite of modifications have not been studied in various publications where the effects of local mass/stiffness modifications on the eigencharacteristics of structures are studied [1–4], it was decided that this special case be investigated.

It was thought that an appropriate method of solution could be to restrain the beam with a spring to counteract the frequency-decreasing effects of mass addition. Another approach could be "dynamic stiffening" of the beam without adding a spring as in reference [5].

Within this framework, the required coefficients of the springs to be placed at certain locations such that the fundamental frequencies will remain the same although there are added masses, are calculated for practically reasonable mass ranges. The results for various supporting conditions are presented as separate tables.

Although it is acknowledged that the contributions of this study do not solve a very complex problem, it is nevertheless thought that the presented tables can be quite helpful for a design engineer who has to solve a problem of this nature.

2. THEORY

The problem can be stated referring to the cantilever beam in Figure 1. Denote the dimensionless frequencies which are defined with respect to the reference frequency

$$\omega_0 = \sqrt{EI/mL^4} \tag{1}$$

of the bare cantilever beam (bending rigidity: EI, mass per unit length: m, length: L), as $\omega_1^*, \omega_2^*, \dots$.

One requires that the fundamental frequency remains unchanged in spite of an additional mass $\bar{m}_l$ placed at $\bar{x}_l$ for some reason. Assume that a spring of stiffness k_k is placed at $\bar{x}_k$ to achieve this.

The dimensionless quantities that are used are defined as follows:

$$\bar{x}_l = \bar{x}_l/L, \quad \bar{x}_k = x_k/L, \quad \bar{m}_l = m_l/mL, \quad \bar{k}_k = k_k/EI/L^3. \tag{2}$$

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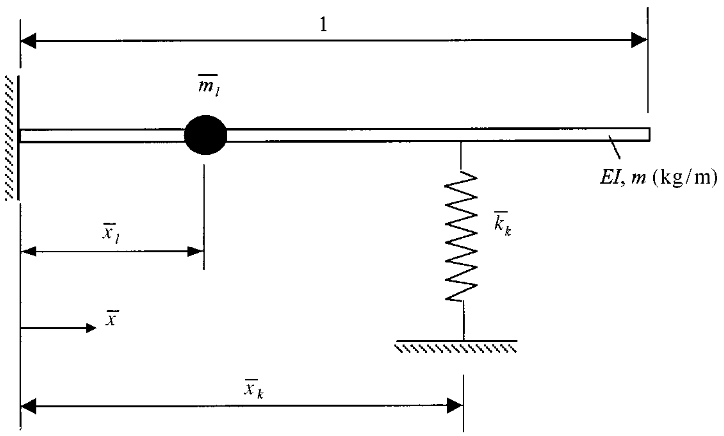

Figure 1. After attaching a point mass $\bar{m}_l$ at $\bar{x}_l$, a restraining spring $\bar{k}_k$ is applied at $\bar{x}_k$ in order to preserve the fundamental frequency of the bare cantilever.

The frequency equation of the system in Figure 1 which carries the mass $\bar{m}_l$ and restrained by the spring $\bar{k}_k$, can be written as [6]

$$\det(\mathbf{K}^* - \omega^{*2}\mathbf{M}^*) = 0, \tag{3}$$

where

$$\omega^* = \omega/\omega_0, \quad \mathbf{K}^* = \bar{\mathbf{B}} + \bar{k}_k \mathbf{a}(\bar{x}_k)\mathbf{a}^{\mathrm{T}}(\bar{\mathbf{x}}_k), \quad \bar{\mathbf{B}} = \mathbf{diag}(\bar{\beta}_k^4), \quad \mathbf{M}^* = \mathbf{I} + \bar{m}_l \mathbf{a}(\bar{x}_l)\mathbf{a}^{\mathrm{T}}(\bar{x}_l),$$

$$\mathbf{a}(\bar{x}) = [a_1(\bar{x}), \ldots, a_n(\bar{x})]^{\mathrm{T}}, \quad \mathbf{a}_k(\bar{x}) = \cosh\bar{\beta}_k\bar{x} - \cos\bar{\beta}_k\bar{x} - \bar{\eta}_k(\sinh\bar{\beta}_k\bar{x} - \sin\bar{\beta}_k\bar{x}),$$

$$\bar{\eta}_k = (\cosh\bar{\beta}_k + \cos\bar{\beta}_k)/(\sinh\bar{\beta}_k + \sin\bar{\beta}_k), \quad \bar{x} = x/L, \tag{4}$$

$$\bar{\beta}_1 = 1{\cdot}87510406,\ \bar{\beta}_2 = 4{\cdot}69409113,\ \bar{\beta}_3 = 7{\cdot}85475748,\ \bar{\beta}_4 = 10{\cdot}99554073$$

$$\bar{\beta}_5 = 14{\cdot}13716839,\ \bar{\beta}_6 = 17{\cdot}27875953,\ \bar{\beta}_7 = 20{\cdot}42035225,\ \bar{\beta}_8 = 23{\cdot}56194490,$$

$$\bar{\beta}_9 = 26{\cdot}70353755,\ \bar{\beta}_{10} = 29{\cdot}84513020, \ldots$$

$\mathbf{I}$ = n-dimensional unit matrix.

Generally, the procedure is to find the dimensionless frequencies ω^* that will satisfy equation (3), given the matrices $\mathbf{K}^*$ and $\mathbf{M}^*$. For the present case, the following problem is considered: It is required that $\omega^* = \omega_1^* = \bar{\beta}_1^2$. The parameters $\bar{x}_l$, $\bar{m}_l$ and therefore the mass matrix $\mathbf{M}^*$ are known. In addition, $\bar{x}_k$, i.e., the location for the spring is chosen. The dimensionless spring constant $\bar{k}_k$ is the parameter to be found, meaning that the equation

$$\det[\mathbf{K}^*(\bar{x}_k, \bar{k}_k) - \omega_1^{*2}\mathbf{M}^*(\bar{x}_l, \bar{m}_l)] = 0 \tag{5}$$

has to be solved with respect to $\bar{k}_k$.

The $\bar{k}_k$ results found with MATLAB for various $\bar{x}_l$, $\bar{m}_l$ and $\bar{x}_k$ values are given in Table 1.

3. NUMERICAL RESULTS

Considering the fact that the tip stiffness of a cantilever beam is $3EI/L^3$, the $\bar{k}_k$ values found from equation (5) are entered in Table 1 after dividing by 3. In other words, the numbers seen in Table 1 are the dimensionless stiffness values with reference to the tip stiffness of the beam.

The following can be observed in Table 1: The numbers of diagonal cells, which correspond to the case of collocated added mass-straining spring, are the same e.g., $\bar{k}_k = 0{\cdot}4120$ for $\bar{m}_l = 0{\cdot}1$. This can be easily explained since the added spring constant is $k_k = 3 \times 0{\cdot}4120 EI/L^3$ and the mass is $m_l = 0{\cdot}1\, mL$. The frequency of this spring–mass subsystem is $\sqrt{k_k/m_l} = 3{\cdot}5157\ \sqrt{EI/mL^4}$ which is equal practically to the fundamental frequency $\bar{\beta}_1^2\omega_0$ of the bare beam, that we required to remain unchanged. The other observations are

- The restraining spring at a certain location becomes stiffer as the mass added at a certain location is increased.
- As the added mass approaches the root, the restraining springs become softer as they approach the free end.
- As the added mass approaches the free end, the restraining springs become stiffer as they approach the root.

When a certain additional mass is shifted towards the free end, the springs to be placed at a certain location, generally becomes stiffer.

It is seen that some of the cells in the Table 1 are empty. This means that the frequency compensation cannot be achieved for the corresponding case, e.g., for $\bar{m}_l = 0{\cdot}25$, $\bar{x}_l = 0{\cdot}8$ and $\bar{x}_k = 0{\cdot}1$.

So far, only cantilevered beam is deal with. The results corresponding to other supporting conditions are shown in Tables 2–4.

3.1. CLAMPED–SIMPLY SUPPORTED

In this case, following changes in equation (4) will become

$$a_k(\bar{x}) = \cosh \bar{\beta}_k \bar{x} - \cos \bar{\beta}_k \bar{x} - \bar{\eta}_k(\sinh \bar{\beta}_k \bar{x} - \sin \bar{\beta}_k \bar{x}),$$
$$\bar{\eta}_k = (\cosh \bar{\beta}_k - \cos \bar{\beta}_k)/(\sinh \bar{\beta}_k - \sin \bar{\beta}_k),$$

$$\bar{\beta}_1 = 3{\cdot}92660231,\ \bar{\beta}_2 = 7{\cdot}06858274,\ \bar{\beta}_3 = 10{\cdot}21017612,\ \bar{\beta}_4 = 13{\cdot}35176877,$$

$$\bar{\beta}_5 = 16{\cdot}49336143,\ \bar{\beta}_6 = 19{\cdot}63495408,\ \bar{\beta}_7 = 22{\cdot}77654673,\ \bar{\beta}_8 = 25{\cdot}91813939,$$

$$\bar{\beta}_9 = 29.05973205,\ \bar{\beta}_{10} = 32{\cdot}20132469,\ \ldots\ .$$

Considering the fact that the midpoint stiffness of a clamped–simply supported beam is $768/7\ EI/L^3$, the $\bar{k}_k$ values found from equation (5) are entered in Table 2 after dividing by the factor 768/7. In other words, the numbers seen in Table 2 are the dimensionless stiffness values with reference to the midpoint stiffness of the beam.

TABLE 1

Necessary values of $\bar{k}_k$ in dependence of $\bar{x}_l$, $\bar{m}_l$ and $\bar{x}_k$: cantilevered beam

$\bar{m}_l$			$\bar{x}_l$ 0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·1	$\bar{x}_k$	0·1	0·4120	5·9832	27·5978	80·7336	188·2008	392·7718	800·9687	1799·9608	6790·1072	
		0·2	0·0284	0·4120	1·8844	5·3908	12·0158	23·1223	40·7841	68·8178	115·6392	204·3846
		0·3	0·0062	0·0902	0·4120	1·1710	2·5752	4·8418	8·2330	13·1344	20·2145	30·8049
		0·4	0·0021	0·0318	0·1453	0·4120	0·9006	1·6750	2·8009	4·3603	6·4809	9·3894
		0·5	0·0010	0·0145	0·0666	0·1890	0·4120	0·7617	1·2615	1·9367	2·8237	3·9846
		0·6	0·0005	0·0079	0·0361	0·1026	0·2236	0·4120	0·6782	1·0317	1·4855	2·0614
		0·7	0·0003	0·0048	0·0220	0·0626	0·1365	0·2513	0·4120	0·6228	0·8887	1·2187
		0·8	0·0002	0·0031	0·0146	0·0416	0·0908	0·1672	0·2738	0·4120	0·5840	0·7935
		0·9	0·0001	0·0022	0·0103	0·0295	0·0644	0·1188	0·1945	0·2922	0·4120	0·5558
		1·0	0·0001	0·0016	0·0077	0·0219	0·0480	0·0887	0·1454	0·2184	0·3072	0·4120
0·25	$\bar{x}_k$	0·1	1·0301	14·9892	70·2109	215·7655	568·2523	1614·1177	13307·6936			
		0·2	0·0710	1·0302	4·7218	13·6794	31·5498	65·2956	133·2236	308·1288	1512·8293	
		0·3	0·0155	0·2258	1·0302	2·9353	6·5344	12·6603	22·8430	40·6445	76·9149	187·2670
		0·4	0·0054	0·0797	0·3637	1·0302	2·2581	4·2529	7·3188	12·0279	19·6705	33·8347
		0·5	0·0025	0·0366	0·1671	0·4733	1·0302	1·9107	3·2059	5·0655	7·7847	12·0312
		0·6	0·0013	0·0198	0·0908	0·2576	0·5601	1·0302	1·7016	2·6249	3·8931	5·6984
		0·7	0·0008	0·0121	0·0554	0·1576	0·3432	0·6297	1·0302	1·5634	2·2644	3·2035
		0·8	0·0005	0·0080	0·0368	0·1049	0·2292	0·4211	0·6864	1·0302	1·4665	2·0251
		0·9	0·0004	0·0057	0·0261	0·0746	0·1634	0·3009	0·4907	0·7328	1·0302	1·3964
		1·0	0·0002	0·0042	0·0194	0·0556	0·1223	0·2262	0·3698	0·5519	0·7708	1·0301
0·5	$\bar{x}_k$	0·1	2·0603	30·0825	144·6719	487·6272	1738·4715					
		0·2	0·1421	2·0603	9·4794	28·0608	68·8701	166·5592	544·9199			
		0·3	0·0311	0·4519	2·0603	5·8959	13·4031	27·4189	55·9215	134·6599	1182·8571	
		0·4	0·0109	0·1596	0·7284	2·0603	4·5382	8·7326	15·8303	29·0639	61·1590	255·9825
		0·5	0·0050	0·0733	0·3355	0·9487	2·0603	3·8421	6·5933	10·9757	18·7861	36·8077
		0·6	0·0027	0·0397	0·1827	0·5185	1·1235	2·0603	3·4238	5·4092	8·4678	13·8350
		0·7	0·0016	0·0242	0·1117	0·3186	0·6925	1·2641	2·0603	3·1476	4·6777	7·0075
		0·8	0·0011	0·0161	0·0744	0·2131	0·4658	0·8521	1·3788	2·0603	2·9548	4·1956
		0·9	0·0007	0·0114	0·0528	0·1520	0·3345	0·6153	0·9960	1·4733	2·0603	2·8158
		1·0	0·0005	0·0084	0·0394	0·11399	0·2524	0·4679	0·7613	1·1240	1·5510	2·0603

		0·1	3·0905	45·2810	223·7812	840·73066	5544·3638					
		0·2	0·2133	3·0905	14·2733	43·1999	113·7035	344·8071				
		0·3	0·0467	0·6785	3·0905	8·8822	20·6327	44·8446	108·1012	588·1316		
		0·4	0·0164	0·2399	1·0943	3·0905	6·8407	13·4578	25·8519	55·0582	205·9634	
0·75	$\bar{x}_k$	0·5	0·0075	0·1102	0·5053	1·4263	3·0905	5·7948	10·1778	17·9609	35·5174	117·3884
		0·6	0·0040	0·0599	0·2758	0·7828	1·6902	3·0905	5·1670	8·3678	13·9202	26·4005
		0·7	0·0024	0·0365	0·1690	0·4831	1·0482	1·9031	3·0905	4·7534	7·2551	11·5983
		0·8	0·0016	0·0242	0·1127	0·3246	0·7102	1·2935	2·0775	3·0905	4·4654	6·5276
		0·9	0·0011	0·0172	0·0802	0·2326	0·5139	0·9439	1·5167	2·2216	3·0905	4·2587
		1·0	0·0008	0·0128	0·0599	0·1750	0·3910	0·7266	1·1762	1·7174	2·3407	3·0905
		0·1	4·1207	60·5857	307·9879	1317·8892						
		0·2	0·2844	4·1207	19·1039	59·1580	168·5726	741·6627				
		0·3	0·0623	0·9054	4·1207	11·8946	28·2521	65·7323	202·6432			
		0·4	0·0219	0·32051	1·4613	4·1207	9·1659	18·4491	37·8247	99·5968		
1	$\bar{x}_k$	0·5	0·0100	0·1474	0·6765	1·9060	4·1207	7·7691	13·9773	26·3438	64.0311	
		0·6	0·0054	0·0801	0·3702	1·0505	2·2603	4·1207	6·9316	11·5175	20·5296	48.3629
		0·7	0·0033	0·0489	0·2273	0·6513	1·4104	2·5469	4·1207	6·3809	10·0139	17·2481
		0·8	0·0022	0·0325	0·1519	0·4397	0·9628	1·7458	2·7824	4·1207	5·9988	9·0399
		0·9	0·0015	0·0230	0·1082	0·3164	0·7023	1·2879	2·0533	2·9779	4·1207	5·7257
		1·0	0·0011	0·0171	0·0810	0·2392	0·5389	1·0042	1·6169	2·3332	3·1402	4·1207
		0·1	8·2415	122·8900	707·1000	8864·4352						
		0·2	0·5694	8·2415	38·8015	132·6723	610·4246					
		0·3	0·1249	1·8169	8·2415	24·2114	63·3366	218·1385				
		0·4	0·0440	0·6459	2·9405	8·2415	18·7006	41·5825	123·8900			
2	$\bar{x}_k$	0·5	0·0202	0·2981	1·3750	3·8464	8·2415	15·8886	31·7641	87·8424		
		0·6	0·0109	0·1625	0·7598	2·1567	4·5748	8·2415	14·2117	26·4543	71·3360	
		0·7	0·0066	0·0994	0·4708	1·3631	2·9282	5·1703	8·2415	13·1187	23·3091	64·0421
		0·8	0·0044	0·0662	0·3171	0·9383	2·0634	3·6691	5·6660	8·2415	12·3710	21·3865
		0·9	0·0031	0·0470	0·2276	0·6885	1·5602	2·8403	4·3758	6·0846	8·2415	11·8473
		1·0	0·0023	0·0350	0·1715	0·5306	1·2454	2·3520	3·6911	5·0494	6·4391	8·2415
		0·1	12·3623	186·9869	1244·7992							
		0·2	0·8550	12·3623	59·1207	226·4903	4833·6004					
		0·3	0·1876	2·7344	12·3623	36·9733	108·0725	960·3815				
		0·4	0·0662	0·9763	4·4379	12·3623	28·6266	71·4439	512·9106			
3	$\bar{x}_k$	0·5	0·0304	0·4523	2·0966	5·8221	12·3623	24·3829	55·1633	395·9541		
		0·6	0·0164	0·2472	1·1703	3·3231	6·9454	12·3623	21·8673	46·5981	407·4662	
		0·7	0·0100	0·1516	0·7320	2·1440	4·5660	7·8737	12·3623	20·2440	41·8145	669·4152

TABLE 1
Continued

$\bar{m}_l$			$\bar{x}_l$									
			0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
		0·8	0·0066	0·1011	0·4975	1·5084	3·3337	5·7991	8·6567	12·3623	19·1526	39·2598
		0·9	0·0047	0·0719	0·3599	1·1322	2·6317	4·7480	7·0240	9·3287	12·3623	18·4072
		1·0	0·0035	0·0537	0·2733	0·8934	2·2121	4·2559	6·4484	8·2510	9·9090	12·3623
		0·1	16·4831	252.9551	2008.4349							
		0·2	1·1411	16·4831	80·0916	350·3712						
		0·3	0·2506	3·6581	16·4831	50·2049	167·0778					
		0·4	0·0885	1·3119	5·9539	16·4831	38·9686	111·4674				
4	$\bar{x}_k$	0·5	0·0406	0·6100	2·8424	7·8342	16·4831	33·2784	87·3290			
		0·6	0·0220	0·3344	1·6037	4·5548	9·3742	16·4831	29·9280	75·2465		
		0·7	0·0134	0·2055	1·0132	3·0048	6·3388	10·6608	16·4831	27·7912	69·3392	
		0·8	0·0089	0·1374	0·6951	2·1666	4·8161	8·1709	11·7602	16·4831	26·3844	67·4409
		0·9	0·0063	0·0979	0·5074	1·6706	4·0079	7·1489	10·0715	12·7196	16·4831	25·4542
		1·0	0·0046	0·0732	0·3886	1·3576	3·6150	7·1496	10·2928	12·0811	13·5635	16·4831
		0·1	20·6039	320·8776	3178·2885							
		0·2	1·4277	20·6039	101·7458	521·5218						
		0·3	0·3139	4·5880	20·6039	63·9326	248·4750					
		0·4	0·1109	1·6528	7·4887	20·6039	49·75336	167·9043				
5	$\bar{x}_k$	0·5	0·0509	0·7713	3·6137	9·8836	20·6039	42·6044	134·3234			
		0·6	0·0276	0·4241	2·0616	5·8574	11·8634	20·6039	38·4270	119·2268		
		0·7	0·0168	0·2614	1·3166	3·9582	8·2638	13·5355	20·6039	35·7991	114·6013	
		0·8	0·0111	0·1751	0·9125	2·9351	6·5687	10·8280	14·9833	20·6039	34·1128	118·4601
		0·9	0·0079	0·1250	0·6728	2·3374	5·8404	10·2623	13·6160	16·2673	20·6039	33·0447
		1·0	0·0058	0·0937	0·5204	1·9724	5·8355	12·0763	16·0252	16·7448	17·4177	20·6039

TABLE 2

As in Table 1: clamped–simply supported

$\bar{m}_l$			$\bar{x}_l$								
			0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
		0·1	0·2166	2·5564	9·9092	25·9035	54·1415	77·4226	53·5987	20·2656	4·2333
		0·2	0·0187	0·2166	0·7651	1·6637	2·6767	3·2019	2·6326	1·3591	0·3451
		0·3	0·0054	0·0628	0·2166	0·4461	0·6751	0·7801	0·6566	0·3623	0·0975
		0·4	0·0026	0·0312	0·1080	0·2166	0·3158	0·3568	0·3020	0·1717	0·0476
0·1	$\bar{x}_k$	0·5	0·0018	0·0219	0·0765	0·1529	0·2166	0·2387	0·2014	0·1164	0·0329
		0·6	0·0017	0·0203	0·0716	0·1443	0·2023	0·2166	0·1802	0·1049	0·0301
		0·7	0·0020	0·0248	0·0887	0·1809	0·2542	0·2674	0·2166	0·1254	0·0364
		0·8	0·0036	0·0435	0·1575	0·3271	0·4639	0·4841	0·3825	0·2166	0·0630
		0·9	0·0126	0·1511	0·5560	1·1839	1·7098	1·7792	1·3751	0·7557	0·2166
		0·1	0·5416	6·5880	31·3876	224·1082				345·8331	13·1405
		0·2	0·0470	0·5416	1·9682	4·8523	10·1948	17·0788	14·0530	5·0403	0·9565
		0·3	0·0136	0·1583	0·5416	1·1466	1·9003	2·4861	2·2084	1·1144	0·2607
		0·4	0·0067	0·0797	0·2738	0·5416	0·8102	0·9772	0·8726	0·4864	0·1249
0·25	$\bar{x}_k$	0·5	0·0047	0·0565	0·1987	0·3891	0·5416	0·6106	0·5375	0·3135	0·0852
		0·6	0·0043	0·0528	0·1911	0·3816	0·5164	0·5416	0·4589	0·2726	0·0771
		0·7	0·0052	0·0651	0·2437	0·5045	0·6841	0·6838	0·5416	0·3176	0·0923
		0·8	0·0092	0·1150	0·4479	0·9787	1·3595	1·3194	0·9775	0·5416	0·1585
		0·9	0·0318	0·4037	1·6549	3·9343	5·7819	5·4750	3·7638	1·9239	0·5416
		0·1	1·0833	13·8898	113·1117						43·9998
		0·2	0·0944	1·0833	4·1358	13·4342	159·8852			51·8643	2·3363
		0·3	0·0274	0·3210	1·0833	2·4053	4·8109	9·1733	10·4043	3·6186	0·5899
		0·4	0·0136	0·1647	0·5602	1·0833	1·6942	2·3243	2·3554	1·2501	0·2721
0·5	$\bar{x}_k$	0·5	0·0095	0·1187	0·4244	0·8023	1·0833	1·2699	1·2106	0·7199	0·1810
		0·6	0·0088	0·1127	0·4299	0·8451	1·0705	1·0833	0·9471	0·5835	0·1606
		0·7	0·0107	0·1413	0·5833	1·2482	1·5675	1·4212	1·0833	0·6487	0·1891

TABLE 2

Continued

$\bar{m}_l$			$\bar{x}_l$								
			0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
		0·8	0·0187	0·2542	1·1628	2·9129	3·8123	3·1054	2·0294	1·0833	0·3197
		0·9	0·0647	0·9116	4·8511	17·4371	28·0475	17·7974	8·9404	3·9687	1·0833
		0·1	1·6250	22·0280	856·2605						202·5753
		0·2	0·1422	1·6250	6·5346	32·7304					4·5000
		0·3	0·0414	0·4882	1·6250	3·7934	9·8284	88·7051		14·4146	1·0186
		0·4	0·0206	0·2555	0·8602	1·6250	2·6623	4·3001	5·4328	2·6228	0·4480
0·75	$\bar{x}_k$	0·5	0·0144	0·1877	0·6830	1·2418	1·6250	1·9838	2·0780	1·2677	0·2895
		0·6	0·0133	0·1815	0·7367	1·4197	1·6666	1·6250	1·4673	0·9416	0·2513
		0·7	0·0163	0·2318	1·0891	2·4542	2·7523	2·2188	1·6250	0·9941	0·2906
		0·8	0·0285	0·4259	2·4840	8·5357	9·5641	5·6587	3·1645	1·6250	0·4838
		0·9	0·0988	1·5701	13·6170			71·2506	16·5092	6·1460	1·6250
		0·1	2·1667	31·1551							
		0·2	0·1905	2·1667	9·2037	116·1340					8·3807
		0·3	0·0557	0·6600	2·1667	5·3318	20·5389				1·5999
		0·4	0·0278	0·3527	1·1746	2·1667	3·7271	7·4790	15·6681	5·8153	0·6621
1	$\bar{x}_k$	0·5	0·0195	0·2646	0·9824	1·7101	2·1667	2·7596	3·2378	2·0460	0·4133
		0·6	0·0180	0·2611	1·1454	2·1509	2·3095	2·1667	2·0229	1·3583	0·3501
		0·7	0·0221	0·3410	1·9230	4·7477	4·4243	3·0843	2·1667	1·3548	0·3972
		0·8	0·0387	0·6432	5·7516	245·0416	38·9377	9·6091	4·3932	2·1667	0·6509
		0·9	0·1341	2·4577	141·0919				28·6265	8·4692	2·1667
		0·1	4·3334	82·3137							
		0·2	0·3877	4·3334	23·7628						
		0·3	0·1151	1·3984	4·3334	13·6126					11·1151

		0·4	0·0580	0·8218	2·6003	4·3334	9·3171				2·3358
2	$\bar{x}_k$	0·5	0·0410	0·6852	2·8674	3·9383	4·3334	6·6746	0·1985	25·8685	1·1519
		0·6	0·0382	0·7627	6·8226	9·4564	5·4812	4·3334	4·6814	4·0400	0·8536
		0·7	0·0470	1·1596			49·8223	7·4335	4·3334	2·9726	0·8828
		0·8	0·0827	2·7360					10·5203	4·3334	1·3497
		0·9	0·2889	16·1632						19·5596	4·3334
		0·1	6·5001	181·8493							
		0·2	0·5919	6·5001	50·2695						
		0·3	0·1785	2·2300	6·5001	28·2245					
		0·4	0·0909	1·4760	4·3672	6·5001	18·6316				14·8442
3	$\bar{x}_k$	0·5	0·0648	1·4574	7·9545	6·9616	6·5001	12·6628			2·8496
		0·6	0·0608	2·1200			10·1086	6·5001	8·3308	11·8162	1·6393
		0·7	0·0752	5·8043				14·0265	6·5001	4·9382	1·4899
		0·8	0·1333						19·6595	6·500	2·1019
		0·9	0·4693							34·7105	6·5001
		0·1	8·6669	459·9254							
		0·2	0·8035	8·6669	113·6635						
		0·3	0·2464	3·1737	8·6669	60·9207					
		0·4	0·1270	2·4520	6·6142	8·6669	37·2527				
4	$\bar{x}_k$	0·5	0·0913	3·3384	70·4374	11·2984	8·6669	22·9638			10·8276
		0·6	0·0863	19·2379			17·4921	8·6669	13·6522	313·9953	3·0370
		0·7	0·1075					25·2030	8·6669	7·3771	2·2707
		0·8	0·1921						34·7564	8·6669	2·9140
		0·9	0·6823							56·6518	8·6669
		0·1	10·8336	5574·4483							
		0·2	1·0229	10·8336	467·0748						
		0·3	0·3191	4·2536	10·8336	199·7805					
		0·4	0·1666	4·0645	9·5680	10·8336	93·0531				
5	$\bar{x}_k$	0·5	0·1209	14·7944		18·0419	10·8336	44·8586			
		0·6	0·1153				31·1387	10·8336	22·1357		6·2177
		0·7	0·1449					48·2895	10·8336	10·4840	3·3121
		0·8	0·2611						64·4531	10·8336	3·7932
		0·9	0·9377							91·2668	10·8336

TABLE 3

As in Table 1: *clamped–clamped*

$\bar{m}_l$			$\bar{x}_l$				
			0·1	0·2	0·3	0·4	0·5
		0·1	0·2607	2·9007	11·1263	30·1292	59·5747
		0·2	0·0243	0·2607	0·8440	1·6771	2·3154
		0·3	0·0078	0·0841	0·2607	0·4742	0·6119
		0·4	0·0044	0·0484	0·1504	0·2607	0·3191
0·1	$\bar{x}_k$	0·5	0·0037	0·0412	0·1307	0·2241	0·2607
		0·6	0·0044	0·0499	0·1623	0·2823	0·3191
		0·7	0·0078	0·0896	0·3032	0·5455	0·6119
		0·8	0·0246	0·2889	1·0488	2·0491	2·3154
		0·9	0·2665	3·3471	15·4462	45·7021	59·577
		0·1	0·6517	7·6790	46·8086		
		0·2	0·0612	0·6517	2·2233	5·5729	12·2230
		0·3	0·0197	0·2137	0·6517	1·2447	1·8583
		0·4	0·0112	0·1261	0·3863	0·6517	0·8332
0·25	$\bar{x}_k$	0·5	0·0094	0·1100	0·3549	0·5808	0·6517
		0·6	0·0113	0·1364	0·4755	0·8062	0·8332
		0·7	0·0200	0·2529	1·0032	1·8934	1·8583
		0·8	0·0631	0·8628	4·5782	14·0453	12·2230
		0·9	0·6894	11·8702			
		0·1	1·3035	17·0301			
		0·2	0·1235	1·3035	4·8837	24·6883	
		0·3	0·0400	0·4393	1·3035	2·7145	5·7885
		0·4	0·0229	0·2708	0·8094	1·3035	1·7990
0·5	$\bar{x}_k$	0·5	0·0194	0·2470	0·8289	1·2374	1·3035
		0·6	0·0233	0·3231	1·3334	2·1134	1·7990
		0·7	0·0414	0·6445	4·3557	10·7364	5·7885
		0·8	0·1315	2·5523			
		0·9	1·4635	78·5014			
		0·1	1·9553	28·6662			
		0·2	0·1869	1·9553	8·1242		
		0·3	0·0611	0·6778	1·9553	4·4766	19·6186
		0·4	0·0351	0·4385	1·2748	1·9553	2·9317
0·75	$\bar{x}_k$	0·5	0·0299	0·4226	1·4937	1·9855	1·9553
		0·6	0·0360	0·5942	3·3443	4·5989	2·9317
		0·7	0·0644	1·3317			19·6186
		0·8	0·2059	7·3485			
		0·9	2·3387				
		0·1	2·6071	43·5413			
		0·2	0·2513	2·6071	12·1564		
		0·3	0·0828	0·9302	2·6071	6·6278	
		0·4	0·0479	0·6351	1·7891	2·6071	4·2787
1	$\bar{x}_k$	0·5	0·0409	0·6556	2·4938	2·8457	2·6071
		0·6	0·0495	1·0237	13·5986	11·1627	4·2787
		0·7	0·0890	2·8523			
		0·8	0·2869	121·5734			
		0·9	3·3364				

Table 3

Continued

$\bar{m}_l$			$\bar{x}_l$				
			0·1	0·2	0·3	0·4	0·5
		0·1	5·2142	196·4535			
		0·2	0·5207	5·2142	47·6147		
		0·3	0·1777	2·1076	5·2142	23·7377	
		0·4	0·1053	1·9399	4·5307	5·2142	13·7661
2	$\bar{x}_k$	0·5	0·0919	3·7816		8·1263	5·2142
		0·6	0·1133				13·7661
		0·7	0·2091				
		0·8	0·7007				
		0·9	9·2645				
		0·1	7·8213				
		0·2	0·8100	7·8213	1710·8802		
		0·3	0·2875	3·6456	7·8213	170·1790	
		0·4	0·1753	6·1545	9·2610	7·8213	52·7690
3	$\bar{x}_k$	0·5	0·1570			21·3037	7·8213
		0·6	0·1986				52·7690
		0·7	0·3796				
		0·8	1·3491				
		0·9	22·7217				
		0·1	10·4284				
		0·2	1·1215	10·4284			
		0·3	0·4158	5·7399	10·4284		
		0·4	0·2624		19·3755	10·4284	
4	$\bar{x}_k$	0·5	0·2432			112·5877	10·4284
		0·6	0·3183				
		0·7	0·6409				
		0·8	2·5107				
		0·9	83·0105				
		0·1	13·0355				
		0·2	1·4580	13·0355			
		0·3	0·5681	8·7590	13·0355		
		0·4	0·3738		56·2100	13·0355	
5	$\bar{x}_k$	0·5	0·3624				13·0355
		0·6	0·4988				
		0·7	1·0918				
		0·8	5·1936				
		0·9					

Table 4

As in Table 1: *simply supported at both ends*

m_l			$\bar{x}_l$				
			0·1	0·2	0·3	0·4	0·5
		0·1	0·2029	0·7381	1·4283	2·0443	2·3424
		0·2	0·0562	0·2029	0·3870	0·5461	0·6206
		0·3	0·0297	0·1075	0·2029	0·2825	0·3182
		0·4	0·0216	0·0783	0·1476	0·2029	0·2260
0·1	$\bar{x}_k$	0·5	0·0196	0·0714	0·1351	0·1847	0·2029
		0·6	0·0217	0·0797	0·1516	0·2075	0·2260
		0·7	0·0300	0·1113	0·2136	0·2933	0·3182
		0·8	0·0571	0·2136	0·4141	0·5726	0·6206
		0·9	0·2075	0·7848	1·5438	2·1578	2·3424
		0·1	0·5073	1·8598	3·7207	5·6485	6·9162
		0·2	0·1407	0·5073	0·9773	1·4251	1·6956
		0·3	0·0747	0·2702	0·5073	0·7139	0·8280
		0·4	0·0544	0·1990	0·3719	0·5073	0·5710
0·25	$\bar{x}_k$	0·5	0·0495	0·1838	0·3465	0·4664	0·5073
		0·6	0·0552	0·2081	0·3987	0·5367	0·5710
		0·7	0·0768	0·2957	0·5793	0·7872	0·8280
		0·8	0·1468	0·5793	1·1706	1·6208	1·6956
		0·9	0·5367	2·1878	4·6212	6·6097	6·9162
		0·1	1·0147	3·7695	8·0015	13·7002	19·8110
		0·2	0·2825	1·0147	1·9880	3·0743	4·0117
		0·3	0·1509	0·5451	1·0147	1·4545	1·7771
		0·4	0·1106	0·4088	0·7538	1·0147	1·1629
0·5	$\bar{x}_k$	0·5	0·1014	0·3863	0·7246	0·9481	1·0147
		0·6	0·1137	0·4496	0·8723	1·1394	1·1629
		0·7	0·1596	0·6604	1·3503	1·7949	1·7771
		0·8	0·3078	1·3503	2·9934	4·1574	4·0117
		0·9	1·1394	5·4147	13·7731	21·1657	19·8110
		0·1	1·5220	5·7311	12·9790	26·1030	52·3372
		0·2	0·4253	1·5220	3·0339	5·0051	7·3653
		0·3	0·2286	0·8250	1·5220	2·2231	2·8760
		0·4	0·1687	0·6305	1·1460	1·5220	1·7770
0·75	$\bar{x}_k$	0·5	0·1557	0·6106	1·1389	1·4461	1·5220
		0·6	0·1759	0·7330	1·4443	1·8210	1·7770
		0·7	0·2490	1·1211	2·4269	3·1308	2·8760
		0·8	0·4855	2·4269	6·2245	8·6914	7·3653
		0·9	1·8210	10·6515	40·5248	79·5897	52·3372
		0·1	2·0293	7·7466	18·8385	47·6897	292·2399
		0·2	0·5691	2·0293	4·1167	7·2963	12·6545
		0·3	0·3078	1·1099	2·0293	3·0214	4·1632
		0·4	0·2286	0·8649	1·5489	2·0293	2·4144
1	$\bar{x}_k$	0·5	0·2126	0·8604	1·5946	1·9610	2·0293
		0·6	0·2421	1·0703	2·1487	2·5980	2·4144
		0·7	0·3460	1·7216	4·0356	4·9862	4·1632
		0·8	0·6823	4·0356	13·5226	19·1144	12·6545
		0·9	2·5980	20·6250	1405·1523		292·2399

TABLE 4

Continued

$\bar{m}_l$			$\bar{x}_l$				
			0·1	0·2	0·3	0·4	0·5
		0·1	4·0587	16·3963	58·3573		
		0·2	1·1551	4·0587	8·8611	23·2859	
		0·3	0·6411	2·3033	4·0587	6·5494	12·6686
		0·4	0·4903	1·9566	3·2774	4·0587	5·2269
2	$\bar{x}_k$	0·5	0·4710	2·2267	3·9895	4·2100	4·0587
		0·6	0·5569	3·4573	8·0075	7·2194	5·2269
		0·7	0·8327	8·7657	710·7681	44·9096	12·6686
		0·8	1·7419	710·7681			
		0·9	7·2194				
		0·1	6·0881	26·1166	194·0415		
		0·2	1·7588	6·0881	14·3885	86·3979	
		0·3	1·0031	3·5897	6·0881	10·7230	39·7113
		0·4	0·7926	3·3775	5·2187	6·0881	8·5446
3	$\bar{x}_k$	0·5	0·7917	4·7307	7·9886	6·8152	6·0881
		0·6	0·9825	13·4724	87·8391	17·7342	8·5446
		0·7	1·5676				39·7113
		0·8	3·6110				
		0·9	17·7342				
		0·1	8·1174	37·1195			
		0·2	2·3810	8·1174	20·9100		
		0·3	1·3977	4·9804	8·1174	15·7371	
		0·4	1·1458	5·3031	7·4145	8·1174	12·5170
4	$\bar{x}_k$	0·5	1·2002	10·8074	16·0156	9·8686	8·1174
		0·6	1·5901			65·2558	12·5170
		0·7	2·8052				
		0·8	7·7903				
		0·9	65·2558				
		0·1	10·1468	49·6768			
		0·2	3·0225	10·1468	28·7206		
		0·3	1·8296	6·4889	10·1468	21·8745	
		0·4	1·5641	8·0604	9·9186	10·1468	17·3595
5	$\bar{x}_k$	0·5	1·7387	47·1367	40·3311	13·4968	10·1468
		0·6	2·5284				17·3595
		0·7	5·3303				
		0·8	25·4962				
		0·9					

3.2. CLAMPED–CLAMPED

In this case, the following changes occur in equation (4):

$$a_k(\bar{x}) = \sin\bar{\beta}_k\bar{x} - \sinh\bar{\beta}_k\bar{x} + \bar{\eta}_k(\cos\bar{\beta}_k\bar{x} - \cosh\bar{\beta}_k\bar{x}),$$

$$\bar{\eta}_k = (\cos\bar{\beta}_k - \cosh\bar{\beta}_k)/(\sin\bar{\beta}_k + \sinh\bar{\beta}_k),$$

$$\bar{\beta}_1 = 4{\cdot}73004074,\ \bar{\beta}_2 = 7{\cdot}85320462,\ \bar{\beta}_3 = 10{\cdot}99560783,\ \bar{\beta}_4 = 14{\cdot}13716549,$$

$$\bar{\beta}_5 = 17{\cdot}27875965,\ \bar{\beta}_6 = 20{\cdot}42035224,\ \bar{\beta}_7 = 23{\cdot}56194490,\ \bar{\beta}_8 = 26{\cdot}70353755,$$

$$\bar{\beta}_9 = 32.98672286,\ \bar{\beta}_{10} = 36{\cdot}12831551, \ldots .$$

Considering the fact that the midpoint stiffness of a clamped–clamped beam is $192EI/L^3$, the $\bar{k}_k$ values found from equation (5) are entered in Table 3 after dividing by the factor 192. In other words, the numbers seen in Table 3 are the dimensionless stiffness values with reference to the midpoint stiffness of the clamped–clamped beam.

Because of the symmetry of the system, it is seen clearly that it is enough to show the $\bar{x}_l$ values between 0·1 and 0·5.

3.3. SIMPLY SUPPORTED AT BOTH ENDS

In this special case, the following changes have to be made in equation (4):

$$\mathbf{K}^* = \bar{\mathbf{B}} + 2\bar{k}_k\mathbf{a}(\bar{x}_k)\mathbf{a}^{\mathrm{T}}(\bar{x}_k), \quad \mathbf{M}^* = \mathbf{I} + 2\bar{m}_l\mathbf{a}(\bar{x}_l)\mathbf{a}^{\mathrm{T}}(\bar{x}_l),$$

$$a_k(\bar{x}) = \sin\bar{\beta}_k\bar{x},\ \bar{\beta}_k = k\pi,\ k = 1, 2, \ldots, 10, \ldots .$$

Considering the fact that the midpoint stiffness of a simply supported beam at both ends is $48EI/L^3$, the $\bar{k}_k$ values found from equation (5) are entered in Table 4 after dividing by the factor 48. In other words, the numbers seen in Table 4 are the dimensionless stiffness values with reference to the midpoint stiffness of the simply supported beam at both ends.

Because of the symmetry of the system, it is seen clearly that it is enough to display the $\bar{x}_l$ values between 0·1 and 0·5.

4. CONCLUSIONS

In this study, we have examined the problem of determining the stiffness coefficient of the spring to be placed at a specified position so that the fundamental frequency of the bending beam subject to various supporting conditions does not change despite the addition of a mass at a predefined position.

The values of the spring coefficients calculated for practically reasonable mass ranges and different supporting conditions are tabulated. It is hoped that this information will be valuable for the interested design engineer.

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